

3-3 Rate of Change and Slope

Then

- You graphed ordered pairs in the coordinate plane.

Now

- Use rate of change to solve problems.
- Find the slope of a line.

Why?

- The Daredevil Drop at Wet 'n Wild Emerald Pointe in Greensboro, North Carolina, is a thrilling ride that drops you 76 feet down a steep water chute. A *rate of change* of the ride might describe the distance a rider has fallen over a length of time.



New Vocabulary
rate of change
slope

Common Core State Standards

Content Standards
F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

F.LE.1a Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

Mathematical Practices
2 Reason abstractly and quantitatively.

1 Rate of Change Rate of change is a ratio that describes, on average, how much one quantity changes with respect to a change in another quantity.

KeyConcept Rate of Change

If x is the independent variable and y is the dependent variable, then

$$\text{rate of change} = \frac{\text{change in } y}{\text{change in } x}$$

Real-World Example 1 Find Rate of Change

ENTERTAINMENT Use the table to find the rate of change. Then explain its meaning.

$$\begin{aligned} \text{rate of change} &= \frac{\text{change in } y}{\text{change in } x} \leftarrow \begin{array}{l} \text{dollars} \\ \text{games} \end{array} \\ &= \frac{\text{change in cost}}{\text{change in number of games}} \\ &= \frac{156 - 78}{4 - 2} \\ &= \frac{78}{2} \text{ or } \frac{39}{1} \end{aligned}$$

The rate of change is $\frac{39}{1}$. This means that each game costs \$39.

Guided Practice

1. REMODELING The table shows how the tiled surface area changes with the number of floor tiles.

- Find the rate of change.
- Explain the meaning of the rate of change.

Number of Floor Tiles	Area of Tiled Surface (in ²)
x	y
3	48
6	96
9	144

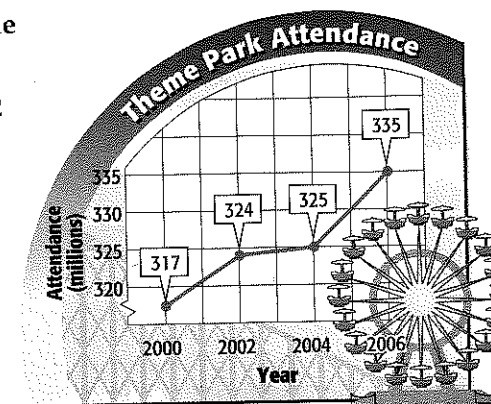
J Silver/Supershot

So far, you have seen rates of change that are *constant*. Many real-world situations involve rates of change that are not constant.

Real-World Example 2 Compare Rates of Change

AMUSEMENT PARKS The graph shows the number of people who visited U.S. theme parks in recent years.

- Find the rates of change for 2000–2002 and 2002–2004.



Source: International Association of Amusement Parks and Attractions

2000–2002:

$$\begin{aligned} \frac{\text{change in attendance}}{\text{change in time}} &= \frac{324 - 317}{2002 - 2000} \leftarrow \begin{array}{l} \text{people} \\ \text{years} \end{array} \quad \text{Substitute.} \\ &= \frac{7}{2} \text{ or } 3.5 \quad \text{Simplify.} \end{aligned}$$

Over this 2-year period, attendance increased by 7 million, for a rate of change of 3.5 million per year.

2002–2004:

$$\begin{aligned} \frac{\text{change in attendance}}{\text{change in time}} &= \frac{325 - 324}{2004 - 2002} \quad \text{Substitute.} \\ &= \frac{1}{2} \text{ or } 0.5 \quad \text{Simplify.} \end{aligned}$$

Over this 2-year period, attendance increased by 1 million, for a rate of change of 0.5 million per year.

- Explain the meaning of the rate of change in each case.

For 2000–2002, on average, 3.5 million more people went to a theme park each year than the last.

For 2002–2004, on average, 0.5 million more people attended theme parks each year than the last.

- How are the different rates of change shown on the graph?

There is a greater vertical change for 2000–2002 than for 2002–2004. Therefore, the section of the graph for 2000–2002 is steeper.

Guided Practice

- Refer to the graph above. Without calculating, find the 2-year period that has the least rate of change. Then calculate to verify your answer.

StudyTip

Reasoning A positive rate of change indicates an increase over time. A negative rate of change indicates that a quantity is decreasing.

A rate of change is constant for a function when the rate of change is the same between any pair of points on the graph of the function. Linear functions have a constant rate of change.

Example 3 Constant Rates of Change

Determine whether each function is linear. Explain.

a.

x	y
1	-6
4	-8
7	-10
10	-12
13	-14

x	y	rate of change
1	-6	$\frac{-8 - (-6)}{4 - 1}$ or $-\frac{2}{3}$
4	-8	$\frac{-10 - (-8)}{7 - 4}$ or $-\frac{2}{3}$
7	-10	$\frac{-12 - (-10)}{10 - 7}$ or $-\frac{2}{3}$
10	-12	$\frac{-14 - (-12)}{13 - 10}$ or $-\frac{2}{3}$

The rate of change is constant. Thus, the function is linear.

b.

x	y
-3	10
-1	12
1	16
3	18
5	22

x	y	rate of change
-3	10	$\frac{12 - 10}{-1 - (-3)}$ or 1
-1	12	$\frac{16 - 12}{1 - (-1)}$ or 2
1	16	$\frac{18 - 16}{3 - 1}$ or 1
3	18	$\frac{22 - 18}{5 - 3}$ or 2

This rate of change is not constant. Thus, the function is not linear.

Guided Practice

3A.

x	y
-3	11
-2	15
-1	19
1	23
2	27

3B.

x	y
12	-4
9	1
6	6
3	11
0	16

2 Find Slope The slope of a nonvertical line is the ratio of the change in the y-coordinates (rise) to the change in the x-coordinates (run) as you move from one point to another.

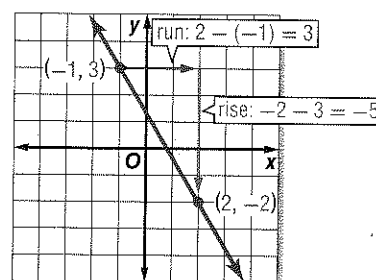
It can be used to describe a rate of change. Slope describes how steep a line is. The greater the absolute value of the slope, the steeper the line.

The graph shows a line that passes through $(-1, 3)$ and $(2, -2)$.

$$\begin{aligned} \text{slope} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{\text{change in } y\text{-coordinates}}{\text{change in } x\text{-coordinates}} \\ &= \frac{-2 - 3}{2 - (-1)} \text{ or } -\frac{5}{3} \end{aligned}$$

So, the slope of the line is $-\frac{5}{3}$.

Because a linear function has a constant rate of change, any two points on a nonvertical line can be used to determine its slope.



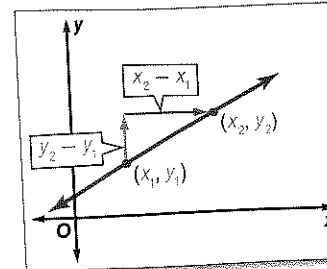
KeyConcept Slope

Words The slope of a nonvertical line is the ratio of the rise to the run.

Symbols The slope m of a nonvertical line through any two points, (x_1, y_1) and (x_2, y_2) , can be found as follows.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \begin{array}{l} \leftarrow \text{change in } y \\ \leftarrow \text{change in } x \end{array}$$

Graph



ReadingMath

Subscripts y_1 is read as *y sub one* and x_2 is read as *x sub two*. The 1 and 2 are subscripts and refer to the first and second point to which the x - and y -values correspond.

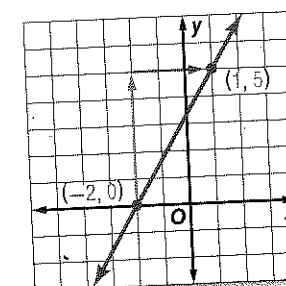
The slope of a line can be positive, negative, zero, or undefined. If the line is not horizontal or vertical, then the slope is either positive or negative.

Example 4 Positive, Negative and Zero Slope

Find the slope of a line that passes through each pair of points.

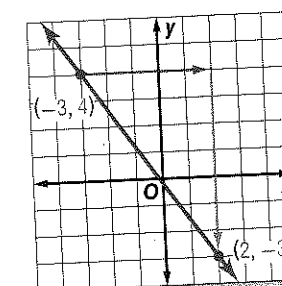
a. $(-2, 0)$ and $(1, 5)$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \frac{\text{rise}}{\text{run}} \\ &= \frac{5 - 0}{1 - (-2)} && (-2, 0) = (x_1, y_1) \text{ and } (1, 5) = (x_2, y_2) \\ &= \frac{5}{3} && \text{Simplify.} \end{aligned}$$



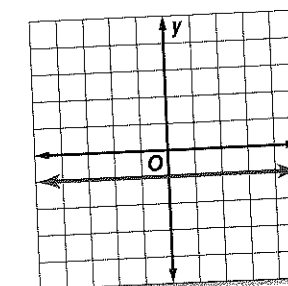
b. $(-3, 4)$ and $(2, -3)$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \frac{\text{rise}}{\text{run}} \\ &= \frac{-3 - 4}{2 - (-3)} && (-3, 4) = (x_1, y_1) \text{ and } (2, -3) = (x_2, y_2) \\ &= -\frac{7}{5} \text{ or } -\frac{7}{5} && \text{Simplify.} \end{aligned}$$



c. $(-3, -1)$ and $(2, -1)$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \frac{\text{rise}}{\text{run}} \\ &= \frac{-1 - (-1)}{2 - (-3)} && \text{Substitute.} \\ &= \frac{0}{5} \text{ or } 0 && \text{Simplify.} \end{aligned}$$



Guided Practice

Find the slope of the line that passes through each pair of points.

- 4A. $(3, 6), (4, 8)$ 4B. $(-4, -2), (0, -2)$ 4C. $(-4, 2), (-2, 10)$
 4D. $(6, 7), (-2, 7)$ 4E. $(-2, 2), (-6, 4)$ 4F. $(4, 3), (-1, 11)$

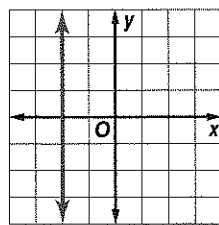
StudyTip

Zero and Undefined Slopes If the change in y -values is 0, then the graph of the line is horizontal. If the change in x -values is 0, then the slope is undefined. This graph is a vertical line.

Example 5 Undefined Slope

Find the slope of the line that passes through $(-2, 4)$ and $(-2, -3)$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{rise} \\ &= \frac{-3 - 4}{-2 - (-2)} && \text{Substitute.} \\ &= \frac{-7}{0} \text{ or undefined} && \text{Simplify.} \end{aligned}$$



GuidedPractice

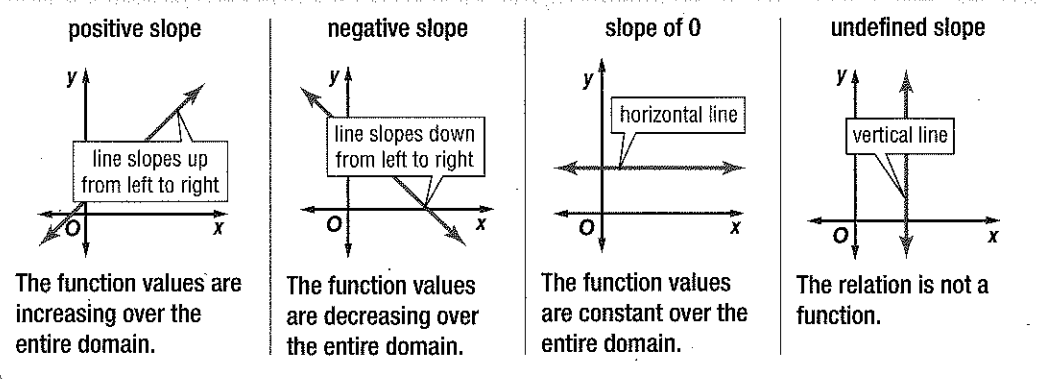
Find the slope of the line that passes through each pair of points.

5A. $(6, 3), (6, 7)$

5B. $(-3, 2), (-3, -1)$

The graphs of lines with different slopes are summarized below.

ConceptSummary Slope



Example 6 Find Coordinates Given the Slope

Find the value of r so that the line through $(1, 4)$ and $(-5, r)$ has a slope of $\frac{1}{3}$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{Slope Formula} \\ \frac{1}{3} &= \frac{r - 4}{-5 - 1} && \text{Let } (1, 4) = (x_1, y_1) \text{ and } (-5, r) = (x_2, y_2). \\ \frac{1}{3} &= \frac{r - 4}{-6} && \text{Subtract.} \\ 3(r - 4) &= 1(-6) && \text{Find the cross products.} \\ 3r - 12 &= -6 && \text{Distributive Property} \\ 3r &= 6 && \text{Add 12 to each side and simplify.} \\ r &= 2 && \text{Divide each side by 3 and simplify.} \end{aligned}$$

So, the line goes through $(-5, 2)$.

GuidedPractice

Find the value of r so the line that passes through each pair of points has the given slope.

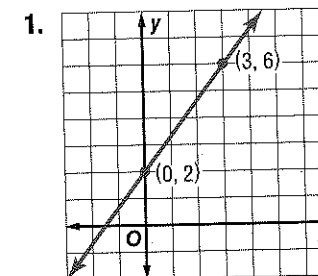
6A. $(-2, 6), (r, -4); m = -5$

6B. $(r, -6), (5, -8); m = -8$

Check Your Understanding

Step-by-Step Solutions begin on page R13.

Example 1 Find the rate of change represented in each table or graph.



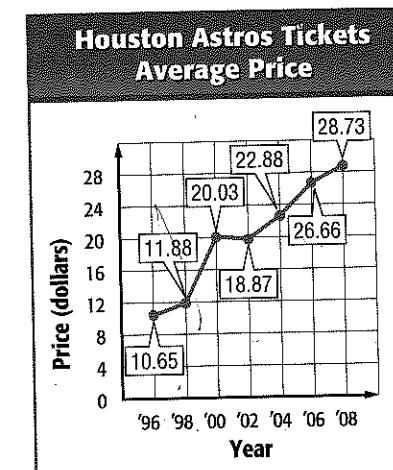
2.

x	y
3	-6
5	2
7	10
9	18
11	26

Example 2

3. **CCSS SENSE-MAKING** Refer to the graph at the right.

- Find the rate of change of prices from 2006 to 2008. Explain the meaning of the rate of change.
- Without calculating, find a two-year period that had a greater rate of change than 2006–2008. Explain.
- Between which years would you guess the new stadium was built? Explain your reasoning.



Source: Team Marketing Report

Example 3 Determine whether each function is linear. Write *yes* or *no*. Explain.

4.

x	-7	-4	-1	2	5
y	5	4	3	2	1

5.

x	8	12	16	20	24
y	7	5	3	0	-2

Examples 4–5 Find the slope of the line that passes through each pair of points.

- $(5, 3), (6, 9)$
- $(-4, 3), (-2, 1)$
- $(6, -2), (8, 3)$
- $(1, 10), (-8, 3)$
- $(-3, 7), (-3, 4)$
- $(5, 2), (-6, 2)$

Example 6 Find the value of r so the line that passes through each pair of points has the given slope.

- $(-4, r), (-8, 3), m = -5$
- $(5, 2), (-7, r), m = \frac{5}{6}$

Practice and Problem Solving

Extra Practice is on page R3.

Example 1 Find the rate of change represented in each table or graph.

14.

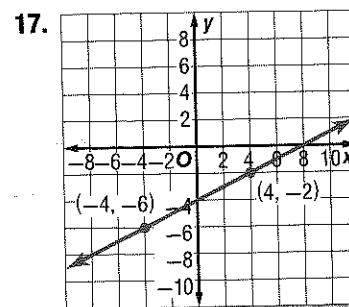
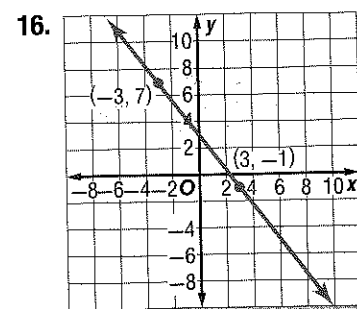
x	y
5	2
10	3
15	4
20	5

15.

x	y
1	15
2	9
3	3
4	-3

Example 1

Find the rate of change represented in each table or graph.



Example 2

18. **SPORTS** What was the annual rate of change from 2004 to 2008 for women participating in collegiate lacrosse? Explain the meaning of the rate of change.

Year	Number of Women
2004	5545
2008	6830

19. **RETAIL** The average retail price in the spring of 2009 for a used car is shown in the table at the right.

Age (years)	Value (\$)
2	17,378
3	16,157

- Write a linear function to model the price of the car with respect to age.
- Interpret the meaning of the slope of the line.
- Assuming a constant rate of change predict the average retail price for a 7-year-old car.

Example 3

Determine whether each function is linear. Write *yes* or *no*. Explain.

20.

x	4	2	0	-2	-4
y	-1	1	3	5	7

21.

x	-7	-5	-3	-1	0
y	11	14	17	20	23

22.

x	-0.2	0	0.2	0.4	0.6
y	0.7	0.4	0.1	0.3	0.6

23.

x	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$
y	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$

Examples 4–5 Find the slope of the line that passes through each pair of points.

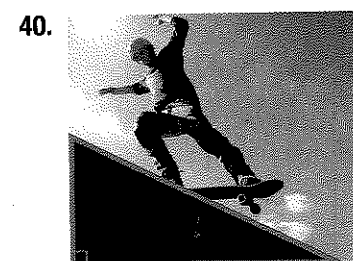
24. (4, 3), (-1, 6) 25. (8, -2), (1, 1) 26. (2, 2), (-2, -2)
 27. (6, -10), (6, 14) 28. (5, -4), (9, -4) 29. (11, 7), (-6, 2)
 30. (-3, 5), (3, 6) 31. (-3, 2), (7, 2) 32. (8, 10), (-4, -6)
 33. (-8, 6), (-8, 4) 34. (-12, 15), (18, -13) 35. (-8, -15), (-2, 5)

Example 6

Find the value of r so the line that passes through each pair of points has the given slope.

36. (12, 10), (-2, r), $m = -4$ 37. (r , -5), (3, 13), $m = 8$
 38. (3, 5), (-3, r), $m = \frac{3}{4}$ 39. (-2, 8), (r , 4), $m = -\frac{1}{2}$

CCSS TOOLS Use a ruler to estimate the slope of each object.



42. **DRIVING** When driving up a certain hill, you rise 15 feet for every 1000 feet you drive forward. What is the slope of the road?

Find the slope of the line that passes through each pair of points.

43.

x	y
4.5	-1
5.3	2

44.

x	y
0.75	1
0.75	-1

45.

x	y
$2\frac{1}{2}$	$-1\frac{1}{2}$
$-\frac{1}{2}$	$\frac{1}{2}$

46. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate why the slope of a line through any two points on that line is constant.

- Visual** Sketch a line ℓ that contains points A , B , A' and B' on a coordinate plane.
- Geometric** Add segments to form right triangles ABC and $A'B'C'$ with right angles at C and C' . Describe \overline{AC} and $\overline{A'C'}$, and \overline{BC} and $\overline{B'C'}$.
- Verbal** How are triangles ABC and $A'B'C'$ related? What does that imply for the slope between any two distinct points on line ℓ ?

47. **BASKETBALL** The table shown below shows the average points per game (PPG) Michael Redd has scored in each of his first 9 seasons with the NBA's Milwaukee Bucks.

Season	1	2	3	4	5	6	7	8	9
PPG	2.2	11.4	15.1	21.7	23.0	25.4	26.7	22.7	21.2

- Make a graph of the data. Connect each pair of adjacent points with a line.
- Use the graph to determine in which period Michael Redd's PPG increased the fastest. Explain your reasoning.
- Discuss the difference in the rate of change from season 1 through season 4, from season 4 through season 7, from season 7 through season 9.

H.O.T. Problems Use Higher-Order Thinking Skills

48. **REASONING** Why does the Slope Formula not work for vertical lines? Explain.

49. **OPEN ENDED** Use what you know about rate of change to describe the function represented by the table.

Time (wk)	Height of Plant (in.)
4	9.0
6	13.5
8	18.0

50. **CHALLENGE** Find the value of d so the line that passes through (a, b) and (c, d) has a slope of $\frac{1}{2}$.

51. **WRITING IN MATH** Explain how the rate of change and slope are related and how to find the slope of a line.

52. **CCSS ARGUMENTS** Kyle and Luna are finding the value of a so the line that passes through $(10, a)$ and $(-2, 8)$ has a slope of $\frac{1}{4}$. Is either of them correct? Explain.

Kyle

$$\frac{2 - 10}{8 - a} = \frac{1}{4}$$

$$4(2 - 10) = 1(8 - a)$$

$$8 - a = -48$$

$$a = 56$$

Luna

$$\frac{8 - a}{-2 - 10} = \frac{1}{4}$$

$$4(8 - a) = 1(-12)$$

$$32 - 4a = -12$$

$$a = 11$$

Image 100/CORBIS, iMar Photographs/Alamy